Sprinkler Bifurcations and Stability

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In dynamical systems, we iterate families of functions. Even the simple family of parabolas of the form $f_c(x) = x^2 + c$, where c is a parameter, has interesting behavior under iteration when one varies c. We look for periodic points (x-values which return to themselves under iteration), and are especially interested in determining how they change as the parameter varies. Typically, a small change in the parameter does not affect the number or type of periodic points. If a change does occur, we call that a bifurcation. It is common for a function to change from 0, to 1, to 2, to 4, to 8 periodic points, and so on. The quadratic family above exhibits such "period-doubling" bifurcations. But is it possible for n new periodic points to appear all at once? In this paper, we introduce examples of functions with this atypical behavior. These in turn lead to results about the stabilities of periodic points near a bifurcation. Our proofs make surprising use of two of the fundamental results of calculus: the Intermediate Value Theorem and the Mean Value Theorem.

This paper grew from two undergraduate research projects on bifurcations in dynamical systems, with students Jon Armel, Missy Larson, Rana Mikkelson, and Dan Wolf. Although much of this work could be considered within the framework of more advanced singularity theory, we choose to present it in this more accessible context.

Generic bifurcations

Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function. Given a point $x_0 \in \mathbb{R}$, the nth *iterate* (where $n \in \mathbb{N}$) of x_0 under f is:

$$f^n(x_0) = \underbrace{f(f(f(\ldots f(x_0))))}_{n \text{ times}} = x_n. \tag{1}$$

The *orbit* of a point, x_0, is the sequence $\{x_0, x_1, x_2, x_3, \ldots\}$. In this paper we focus on *fixed points*($f(x_0) = x_0$) and *periodic points* ($f^n(x_0) = x_0$ for some n). These points can be classified into three groups: attracting, repelling, and neutral. Using Devaney's definition **[2]**, a periodic point x_0 for f of period n is said to be *attracting* or *stable* if $|(f^n)'(x_0)| < 1$, *repelling* or *unstable* if $|(f^n)'(x_0)| > 1$, and *neutral* if $|(f^n)'(x_0)| = 1$. It is shown in **[2]** that around an attracting fixed point x_0 there is an interval in which orbits move closer to x_0 under iteration, and around a repelling fixed point there is an interval in which orbits (other than the fixed point) eventually leave under iteration. For a neutral fixed point a variety of behaviors can occur.

Bifurcations are places where the number and/or stability of periodic points change as the parameter, c, varies. A useful way to visualize bifurcations is with a *bifurcation diagram*, a graph of the x-values of both attracting and repelling periodic points versus c. As an example, Figure 1 shows the points of period two for $x^2 + c$. Note that showing the points of period two also shows all fixed points. (In our bifurcation diagrams, solid lines represent attracting points and dashed lines represent repelling points. For more about bifurcation diagrams, see [**4**].)

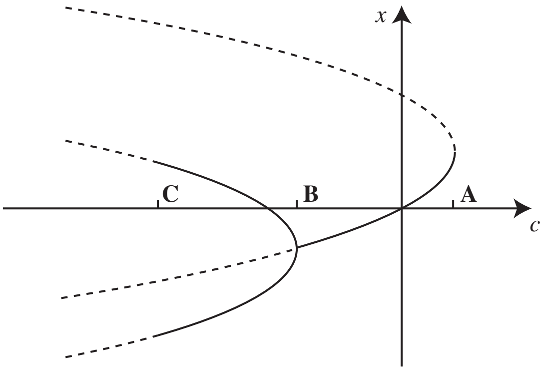

Figure 1. Bifurcation diagram for $x^2 + c$. From right to left: at **A** a new fixed point appears; at **B**, the fixed point changes stability, and simultaneously splits into a 2-cycle. At **C**, the 2-cycle changes stability and a 4-cycle (not shown) appears.

The most common bifurcations are period-doubling and tangent. A *period-doubling bifurcation* is one in which at some value of c a fixed point changes its stability *and* gives rise to a two-cycle which retains the original stability. This occurs in Figure 1 at **B**. Note that in a period doubling bifurcation the fixed point that splits doesn't disappear; it remains, but changes stability. A *tangent bifurcation* (or *saddle-node bifurcation*) is one in which for values of $c < c_0$ there are no fixed points. Then for some particular value, c_0, there is one neutral fixed point, and for any $c > c_0$ there are two fixed points, initially one of which is attracting and the other repelling. It is also possible for these inequalities to be reversed, as in Figure 1 at **A**. These two bifurcations are called *generic* as they will typically persist under perturbation of the family.

Atypical bifurcations

All other bifurcations are *atypical*. Two types commonly discussed in the literature are the pitchfork and the transcritical. A *pitchfork bifurcation* is one in which a single fixed

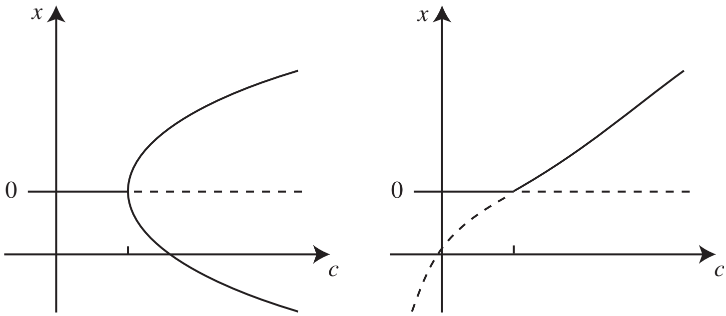

Figure 2. A pitchfork bifurcation (left) and a transcritical bifurcation (right).

point splits as c increases (or decreases) into three fixed points of alternating stability (see Figure 2). A *transcritical bifurcation* is one in which two periodic points of the same period merge into one, then split apart again. Other than at the bifurcation value, one of these periodic points is always attracting and the other is always repelling (see Figure 2).

While examining atypical bifurcations, we discovered a new type which we call a *sprinkler* bifurcation. This is similar to a tangent bifurcation but instead of two, any number of curves of new fixed points emanate from a single point. Specifically, for any natural number n and any x_0 and c_0 in $\mathbb{R}$, there exists a family of smooth functions $f_c : \mathbb{R} \to \mathbb{R}$ for which there are no fixed points for $c < c_0$, one fixed point when $c = c_0$, and n fixed points when $c > c_0$. (See Figure 3. The line with long dashes represents neutral points.)

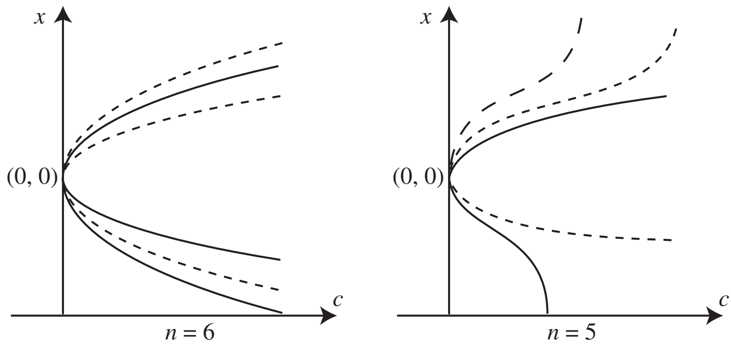

Figure 3. A sprinkler bifurcation for $n = 6$ (left) and $n = 5$ (right).

There are many ways to create examples. The ones that follow were created by the students. In each of the cases, the bifurcation is at $c = 0$, $x = 0$.

Case 1. If n is even, then

$$f_c(x) = x + (x^2 - c)(x^2 - 2c) \cdots \left(x^2 - \frac{n}{2}c\right) \tag{2}$$

will have the desired number of fixed points.

Case 2. If n is odd, $n \geq 3$, let

$$p(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0. \end{cases} \quad (3)$$

As is well-known (see [**3**, pp. 111–112]), $p(x)$ is a smooth function, and its derivatives of all orders at $x = 0$ are zero. Then the piecewise function $h(x)$ below is also smooth:

$$h(x) = \begin{cases} 2e^{-1/x^2} & x < 0 \\ e^{-1/x^2} & x > 0 \\ 0 & x = 0. \end{cases} \quad (4)$$

To construct a function with a three-stream sprinkler bifurcation, we let

$$f_c(x) = x + (h(x) - c)(p(x) - c). \quad (5)$$

For odd order $n \geq 5$, we simply multiply the second term of $f_c(x)$ by $\frac{n-3}{2}$ terms of the form $(x^2 - kc)$ with $k \in \mathbb{N}$, each yielding two new solutions for $c > 0$. For example the bifurcation diagram of

$$x + (h(x) - c)(p(x) - c)(x^2 - c)(x^2 - 2c) \quad (6)$$

has seven new streams of fixed points emanating from the origin.

Case 3. The case $n = 1$ is special. Let

$$g(c) = \begin{cases} e^{-1/c^2} & c < 0 \\ 0 & c \geq 0, \end{cases} \quad (7)$$

and let $f_c(x) = x + x^2 + g(c)$. When $c \geq 0$, f reduces to $x^2 + x$, which has one fixed point at $x = 0$. When $c < 0$, the fixed point equation reduces to a quadratic with non-real roots. Therefore f_c has no fixed points for $c < 0$.

Using similar functions we created *double sprinkler* bifurcations such as the one in Figure 4.

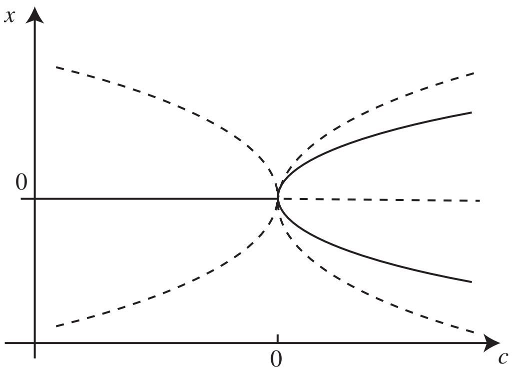

Figure 4. A double sprinkler bifurcation for $m = 3$, $n = 5$.

The Alternating Stability Theorem

Now that we had many examples of bifurcations to explore, we noticed a number of patterns. For example, when an odd number of new fixed points appeared in a bifurcation, one of the new fixed points was neutral. Eventually we could prove some general properties of bifurcations of fixed points. To describe these, further background is required.

We say a fixed point x_0 of f_{c_0} is *continuable* if the only fixed points of f_c near (c_0, x_0) lie on the graph of a continuous function of c in the $c - x$ plane which contains (c_0, x_0). It is shown in [**1**, pp. 462–463] that if x_0 is a fixed point of f_{c_0} and $f'_{c_0}(x_0) \neq 1$, then (c_0, x_0) is continuable. This implies that fixed point bifurcations in which new fixed points appear occur *only* when the derivative at the fixed point is one.

Notice in the transcritical bifurcation (Figure 2), the even sprinkler bifurcation (left side of Figure 3), and the double sprinkler bifurcation (Figure 4) that the stabilities of the fixed points alternate near the bifurcation. We were able to verify that in *any* bifurcation of non-neutral fixed points where a group of curves meet at a single point (c_0, x_0), the stabilities of the curves must alternate for c near c_0.

The Alternating Stability Theorem. *Let n be an integer, with $n \geq 2$. Consider any bifurcation where n curves of non-neutral fixed points intersect at a point (c_0, x_0) in the c-x plane. Then there exists an interval with c_0 as an endpoint in which the stabilities of the curves alternate.*

Proof. Recall that at a bifurcation point we must have $f'_{c_0}(x_0) = 1$. Therefore there is a neighborhood of (c_0, x_0) on which $f'_c(x)$ is positive. We use proof by contradiction and suppose that for a given c in this neighborhood two adjacent fixed points of f_c, a and b, have the same stability. We argue that there must exist a third fixed point between them. Assume that both a and b are attracting. (The argument for repelling fixed points is analogous.) Since the derivatives at a and b are positive, this leads to a graph of f_c like Figure 5. It seems clear from this graph that the smooth function f_c must have a fixed point between a and b. The details of the formal argument of this fact are interesting, and rely on two of the most important theorems from calculus, the Intermediate Value Theorem (IVT) and the Mean Value Theorem (MVT).

Suppose $0 < f'_c(a) < 1$ and $0 < f'_c(b) < 1$ (as in Figure 5). The MVT implies that f_c crosses the line $y = x$ from above to below at a as follows. Since f'_c is contin-

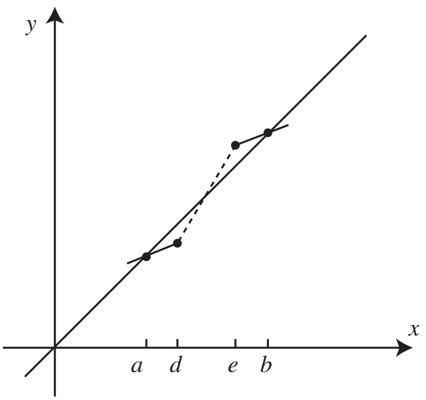

Figure 5. The line $y = x$ contains attracting fixed points a and b; d and e are found using the MVT.

uous, there is a neighborhood of a, say $I = (a - \delta, a + \delta)$, such that $0 < f_c'(x) < 1$ for all $x \in I$. Consider an $x \in I$. By the MVT, there exists z between a and x such that $f_c(x) - f_c(a) = f_c'(z)(x - a)$. If $x < a$, then since $0 < f_c'(z) < 1$, we have $f_c(x) - f_c(a) > x - a$, so $f_c(x) - x > f_c(a) - a = a - a = 0$. If $x > a$, then $f_c(x) - f_c(a) < x - a$, so $f_c(x) - x < 0$. The same is true at b. Therefore, there is a positive $\epsilon < \frac{b-a}{2}$ such that there exists some $d \in (a, a + \epsilon)$ and $e \in (b - \epsilon, b)$ with the property that $f_c(d) - d < 0$ and $f_c(e) - e > 0$. So by the IVT there is an x between d and e such that $f_c(x) - x = 0$. Hence x is a fixed point between a and b, contradicting our assumption that a and b are consecutive fixed points. ■

Note that this theorem applies to the sprinkler bifurcation, in which n new fixed points emanate from a single fixed point, as the single fixed point is a point at which curves of fixed points intersect. This theorem also extends to periodic points, as these are fixed points of the function f_c^p for some p.

The High-Low Stability Theorem

We can further describe the behavior of periodic points near certain bifurcations. The points not only alternate stability, but the smallest and greatest periodic points must retain their stability after the bifurcation.

The High-Low Stability Theorem. *Consider a bifurcation where m non-neutral curves of fixed points intersect in the c-x plane at a point (c_0, x_0), and then split into n curves. Suppose there is a rectangular neighborhood around (c_0, x_0) in which these are the only fixed points. Then the curve consisting of the smallest fixed points has the same stability both before and after the intersection, as does the curve consisting of the largest.*

Proof. We prove the result for the curve consisting of the smallest fixed points. (The case for the largest follows similarly.) For each c value, denote the least of the fixed points by x_c. Using proof by contradiction again, we assume (without loss of generality) that (near c_0) for $c < c_0$, x_c is an attracting fixed point and for $c > c_0$, x_c is a repelling fixed point.

Because (c_0, x_0) is a bifurcation, $f_{c_0}'(x_0) = 1$. We construct a rectangular neighborhood $N = \{(c, x) \mid |c - c_0| < \beta, |x - x_0| < \epsilon\}$ which satisfies the following conditions:

- For all (c, x) in N, $\frac{1}{2} \leq f_c'(x) \leq \frac{3}{2}$,
- The curve x_c (which can be shown to be continuous) leaves the sides of the neighborhood as in Figure 6.

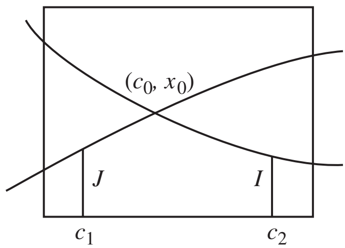

Figure 6.

The first condition can be achieved because $f'_{c_0}(x_0) = 1$, and the second can be achieved by shrinking the width of the rectangle as necessary.

Let $c_1 \in (c_0 - \beta, c_0)$. By assumption $\frac{1}{2} < f'_{c_1}(x_{c_1}) < 1$. Let $J = (x_0 - \epsilon, x_{c_1})$. Then by the same argument as in The Alternating Stability Theorem, there exists an $x' \in J$ with $f_{c_1}(x') > x'$. Since x_{c_1} is the smallest fixed point in N for f_{c_1}, we must have that $f_{c_1}(x) > x$ for all $x \in J$. (If there were a point $x_i \in J$ with $f_{c_1}(x_i) \leq x_i$, there would have to be a fixed point in J less than x_{c_1}, which is a contradiction.) Let $c_2 \in (c_0, c_0 + \beta)$. By assumption $f'_{c_2}(x_{c_2}) > 1$. Let $I = (x_0 - \epsilon, x_{c_2})$. As above we conclude that $f_{c_2}(x) < x$ for all $x \in I$.

Since I and J are open intervals that share a left endpoint, $I \cap J \neq \emptyset$. Thus, we can pick $x^* \in I \cap J$ so x^* is less than the smallest fixed point in N between c_1 and c_2. Then since $x^* \in J$, $f_{c_1}(x^*) > x^*$, and because $x^* \in I$, $f_{c_2}(x^*) < x^*$. Now $f_c(x^*)$ is a continuous function of c, so by the Intermediate Value Theorem there is some $c^* \in (c_1, c_2)$ such that $f_{c^*}(x^*) = x^*$. But x_{c^*} is the *least* fixed point in $(c_0 - \beta, c_0 + \beta)$ by assumption, and x^* is a fixed point less than x_{c^*}, so we have a contradiction. Therefore the least fixed point cannot switch stability when curves of fixed points intersect, and analogously neither can the greatest. ∎

This theorem applies in situations like a pitchfork bifurcation, where there is one fixed point before the bifurcation and three after. This result can also be applied to periodic points.

Neutrality

The above results require non-neutral periodic points in a neighborhood of the bifurcation. We now have the necessary theorems to discuss the point about neutrality raised earlier.

The Odd Fixed Points Theorem. *If the difference in the number of fixed points before and after a bifurcation at a point* (c_0, x_0) *is odd, there exists a one-sided neighborhood about* (c_0, x_0) *in which at least one of the fixed points is neutral.*

Proof. Suppose that the bifurcation occurs at the point $(0, 0)$ and that for $c < 0$, f_c has m fixed points, for $c = 0$, f_c has one fixed point, and for $c > 0$, f_c has $m + n$ fixed points where n is odd. There are two cases, when $m \neq 0$ and when $m = 0$. The first follows easily from the The High-Low Stability Theorem and the The Alternating Stability Theorem.

Suppose then that $m = 0$. Our goal is to construct a function g_c that has the same fixed points (with the same stabilities) as f_c for $c > 0$, but that has two fixed points for $c < 0$, and then appeal to the case $m \neq 0$.

Define g_c as follows.

$$g_c(x) = f_c(x)(x^2 + c) - x^3 - (c - 1)x. \tag{8}$$

Suppose that x_0 is a fixed point for g_c. Then $f_c(x_0)(x_0^2 + c) - x_0^3 - cx_0 + x_0 = x_0$, which simplifies to

$$(f_c(x_0) - x_0)(x_0^2 + c) = 0. \tag{9}$$

When $c > 0$ the only solutions to this equation are the fixed points of f_c, and when $c < 0$ since $f_c(x) = x$ has *no* solutions by assumption, the equation has exactly two solutions, $\pm\sqrt{-c}$. Thus g_c has all of the fixed points of f_c and two new fixed points when $c < 0$.

We must also show that when $c > 0$, the fixed points of f_c have the same stability when they are considered as fixed points of g_c. To do this, we calculate the derivative of g_c.

$$g_c'(x) = f_c'(x)(x^2 + c) + 2xf_c(x) - 3x^2 - c + 1 \tag{10}$$

We only need evaluate this at fixed points of f_c, so we can substitute $f_c(x) = x$. The equation simplifies to

$$g_c'(x) - 1 = (f_c'(x) - 1)(x^2 + c) \tag{11}$$

Note that $x^2 + c$ is positive; so if $f_c'(x) = 1$, then $g_c'(x) = 1$; if $f_c'(x) > 1$, then $g_c'(x) > 1$; and if $f_c'(x) < 1$, then $g_c'(x) < 1$. Thus, the stability of fixed points for f_c is exactly the same as for g_c. We can now apply the case when $m \neq 0$ to g_c to show it has a curve of neutral fixed points for $c > 0$, therefore f_c has a curve of neutral fixed points for $c > 0$. ■

Conclusion

Our functions must be smooth in both x and c. For example, consider the family $f_c(x) = 2|x| + c$, which is continuous, but not differentiable, in x. The bifurcation diagram shows many new periodic points being born at $c = 0$, but all of these new periodic points are repelling. Clearly the alternating fixed point theorem does not hold in this case.

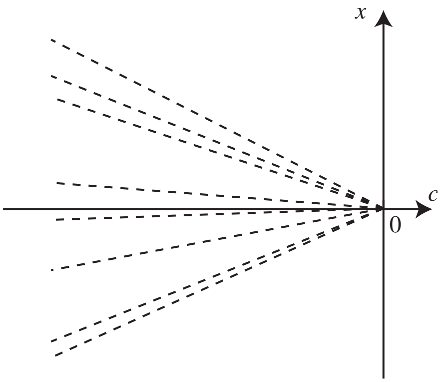

Figure 7. The bifurcation diagram for $f_c(x) = 2|x| + c$.

The above theorems help to characterize the stability of periodic points near bifurcations. As a consequence of these theorems, we understand the stability changes that must occur in pitchfork and transcritical bifurcations. In the future, we hope to consider the existence and the stability of bifurcations where points of different periods meet. Can curves of period two and period three points intersect? Can new points of periods four and five emanate from a single point? If so, what can be said about the stability of these periodic points near the bifurcation? In addition, what can we conclude if we allow the possibility of neutral periodic points near a bifurcation?

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Summary. After discussing common bifurcations of a one-parameter family of single variable functions, we introduce sprinkler bifurcations, in which any number of new fixed points emanate from a single point. Based on observations of these and other bifurcations, we then prove a number of general results about the stability of fixed points near a bifurcation point.

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A Little Love Story

She reads her Calculus text:

Given an epsilon
do si do
find a delta
if you can
Approach
oh so close
Delta on the domain
pursues epsilon on the range
along lazy eight lane
ad infinitum

She begins to doze:

Delta lassoes epsilon
they get married and go live
on the one-over-ex-squared ranch
with acreage they can paint
but never walk around
not enough fence in the universe
to contain it
but enough paint to cover it
strange, so strange

She dreams:

Their herd of discrete cattle
roam the infinite range
bounded by zero below
with domain greater than one
heading off into the horizon
they live happily
ad infinitum.

—Bonnie Shulman (bshulman@bates.edu) Bates College